

WEIGHTED LEAST SQUARES

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WEIGHTED LEAST SQUARES (WLS)

Primary Disciplinary Field(s): Statistics, Econometrics, Data Science, Regression Analysis

1. Core Definition and Relationship to OLS

Weighted Least Squares (WLS) is an advanced regression technique used primarily in statistics and econometrics. It functions as a sophisticated modification of the foundational **Ordinary Least Squares (OLS)** method. While OLS seeks to minimize the sum of squared residuals, treating every observation with equal importance, WLS adapts this minimization objective by incorporating weights that reflect the differential precision or reliability of the individual data points. Fundamentally, WLS introduces a matrix of weights into the standard OLS estimating equation, allowing the model to prioritize or downplay the influence of specific observations based on their known or estimated variance structure. The core definition dictates that WLS is deployed when the standard OLS assumptions, particularly those relating to the structure of the error term, are violated, thereby providing a more efficient and unbiased estimator under specific, non-ideal data conditions.

The mathematical formulation of WLS involves transforming the data or the error terms such that the errors in the transformed model satisfy the classical OLS assumptions. This transformation effectively scales the influence of each observation. Observations associated with smaller variances--meaning they are considered more reliable or precise--are assigned larger weights, thus exerting a greater pull on the fitted regression line. Conversely, observations exhibiting higher variance--often indicating greater uncertainty or noise--receive smaller weights, minimizing their influence on the parameter estimates. This meticulous allocation of importance ensures that the derived coefficients are optimized for the complex distribution of the data, leading to statistical inferences that are robust and trustworthy when heterogeneity is a factor.

The necessity of moving beyond OLS and adopting WLS arises directly from the statistical requirement for best linear unbiased estimators (BLUE). Under the ideal conditions of the **Gauss-Markov theorem**, OLS estimators are BLUE. However, when these conditions, particularly the requirement for **homoscedasticity** (constant variance across all observations), are not met, OLS remains unbiased but loses its "best" quality; that is, it is no longer the most efficient estimator. In the presence of **heteroscedasticity**, OLS standard errors become inconsistent, potentially leading to erroneous hypothesis testing and misleading confidence intervals. WLS corrects for this inefficiency by incorporating the known structure of the variance into the estimation process, thereby restoring the desirable statistical properties of the estimators.

2. The Problem of Heteroscedasticity

WLS is principally employed as a remedy for **heteroscedasticity**, a statistical phenomenon wherein the variance of the error term in a regression model is not constant across all levels of the independent variables. This violation of the homoscedasticity assumption is common in real-world data, particularly in cross-sectional studies where observations often represent entities of vastly different sizes or scales, such as firms, countries, or income groups. For instance, in economic modeling of consumer spending, the variability in spending habits might be much larger for high-income households than for low-income households; the error variance is thus structurally related to the level of income, creating a clear case of heteroscedasticity, which directly violates the core assumption that the variance of the residuals is constant.

The presence of heteroscedasticity has significant implications for the reliability of OLS results. While the estimated regression coefficients (β) remain unbiased, the calculated variance of these coefficients becomes biased and inconsistent. If the variance is increasing with the predictor variable (a common scenario), the OLS estimator tends to underestimate the true standard errors, leading to inflated t-statistics and F-statistics. This can result in Type I errors, where researchers incorrectly reject a true null hypothesis, believing coefficients to be statistically significant when they are not. Conversely, if the heteroscedasticity pattern leads to an overestimation of the standard errors, the model might fail to detect genuinely significant relationships (Type II errors), thus undermining the inferential power of the analysis.

Addressing heteroscedasticity is crucial for achieving statistically valid inference. Methods such as robust standard errors (e.g., **White standard errors**) offer an alternative approach by correcting the standard errors post-estimation without changing the OLS coefficients themselves. However, WLS takes a more fundamental approach: it transforms the data structure *before* estimation, aiming not only to correct the standard errors but also to produce more efficient (lower variance) estimates of the regression coefficients themselves. This efficiency gain makes WLS the preferred method when the precise structure of the heteroscedasticity is known or can be reliably modeled through theoretical justification or empirical evidence gathered during the diagnostic phase.

3. Theoretical Framework of Weighted Least Squares

The theoretical foundation of WLS lies in the **generalized least squares (GLS)** framework. GLS is the overarching method for estimation in linear models when the errors are non-spherical--meaning they are either correlated (autocorrelation) or have non-constant variance (heteroscedasticity). WLS is, in essence, a specialized application of GLS where the non-spherical nature of the errors is purely due to heteroscedasticity, assuming no autocorrelation is present. The general form of the OLS estimator minimizes the sum of squared residuals ($\sum e_i^2$). In contrast, WLS minimizes the weighted sum of squared residuals, $\sum_{i=1}^n w_i e_i^2$, where w_i is the weight assigned to the i -th observation. This adjustment fundamentally changes the objective function to account for the reliability differences across the sample.

Mathematically, the WLS estimator ($\hat{\beta}_{\text{WLS}}$) is derived from the matrix representation: $\hat{\beta}_{\text{WLS}} = (X'WX)^{-1}X'WY$, where X is the matrix of predictors, Y is the vector of the dependent variable, and W is the diagonal matrix of weights. The key transformation is the matrix W . If W were the identity matrix, the equation would revert to the standard OLS formula. In WLS, the diagonal elements of W are defined as the inverse of the variance of the error term for that observation ($w_i = 1 / \sigma_i^2$). By using the inverse variance as the weight, WLS ensures that observations with smaller residual variance (higher reliability) contribute more significantly to the estimation of the regression parameters, thereby achieving the minimum variance linear unbiased estimator (MVLUE) property under the strict assumption that the weights are known correctly.

The efficacy of the WLS method is entirely dependent on the accurate specification of the weights. If the weights are chosen incorrectly--that is, if the assumed variance structure does not match the true underlying variance structure--the resulting WLS estimator will be less efficient than the true GLS estimator, and potentially even less efficient than the original OLS estimator. This critical dependence emphasizes the need for careful diagnostic testing and modeling of the error variance before applying WLS. When the variance function is truly known (e.g., in controlled experimental settings where measurement error variance is known precisely from instrument specifications), WLS yields the most efficient estimates possible, maximizing the utility derived from the available data.

4. Calculation and Implementation of Weights

The practical implementation of WLS centers around determining the appropriate weights, w_i . In academic or highly controlled experimental settings, the weights might be exactly known by design. For instance, if data points are collected at varying levels of precision, or if the dependent variable is an average based on different sample sizes (n_i), the variance of the average is σ^2/n_i , and thus the optimal WLS weights should be proportional to n_i . Similarly, if the variance is known to be proportional to some power of the independent variable, X (e.g., $\sigma_i^2 \propto X_i^2$), the weight w_i would be $1/X_i^2$.

However, in most observational studies, the variances (σ_i^2) are unknown and must be estimated from the data itself. This leads to the procedure known as **Feasible Weighted Least Squares (FWLS)**. The FWLS procedure typically involves a multi-step process. The first step uses OLS to fit the initial model and obtain the residuals (e_i). These residuals are then used as proxies to estimate the variance function. For example, one might regress a transformation of the squared OLS residuals ($\log(e_i^2)$ or e_i^2) on the predictor variables (or a transformation thereof, like X_i or X_i^2) to model the systematic relationship between the variance and the predictors.

Once the variance function $\hat{\sigma}_i^2$ is estimated in the intermediate stage, the estimated weights are calculated as $\hat{w}_i = 1 / \hat{\sigma}_i^2$. These estimated weights are then used in the WLS formula to obtain the final, more efficient regression estimates. It is crucial to iterate this process if the estimated weights from the initial OLS residuals are deemed insufficiently accurate, although often one well-specified iteration provides substantial improvement over OLS. The success of FWLS hinges entirely on the accurate specification of the functional form relating the variance to the predictors. If this form is modeled incorrectly, the resulting parameter estimates may actually be less reliable than the original OLS estimates, highlighting the complexity inherent in this methodological choice.

5. Assumptions and Conditions for Use

While WLS relaxes the strict homoscedasticity assumption of OLS, it still relies heavily on the other core assumptions of the general linear model. The paramount assumption that remains critical is the correct specification of the functional form of the model--meaning the relationship between the predictors and the response variable is truly linear. If the model is misspecified (e.g., omitting relevant variables or incorrectly modeling a non-linear relationship as linear), WLS cannot correct for the resulting bias in the coefficient estimates, regardless of how meticulously the weights are chosen. Furthermore, WLS retains the assumption that the errors are independent across observations (no autocorrelation), and that the errors are normally distributed for the strict validity of small-sample inference (though this latter requirement is mitigated for large samples due to the Central Limit Theorem).

The most defining condition for the appropriate use of WLS is the knowledge or reliable estimation of the variance structure. If the variance structure is unknown or cannot be modeled accurately, WLS should be avoided, as incorrectly specified weights can lead to estimates that are statistically less efficient, and potentially more biased, compared to OLS or the use of Heteroscedasticity-Consistent (HC) standard errors. Therefore, researchers must conduct rigorous diagnostic checks, such as visual inspection of scatter plots of residuals against fitted values or independent variables, and employ formal statistical tests like the **Breusch-Pagan test** or the White test, to confirm both the presence and the precise nature of the heteroscedasticity before proceeding with WLS.

In essence, WLS is only conditionally superior to OLS. It demands that the researcher possesses sufficient theoretical or empirical knowledge to accurately specify the weighting matrix W . If the primary goal is solely to achieve unbiased and consistent standard errors without modifying the coefficient estimates, the HC methods are often preferred due to their robustness to unknown forms of heteroscedasticity. However, if attaining efficiency gains in the parameter estimates is the highest priority and the variance function is well-understood or theoretically justified, WLS remains the statistically optimal technique, restoring the BLUE property to the estimators.

6. Advantages and Limitations

The principal advantage of WLS over OLS in the presence of known heteroscedasticity is its superior efficiency. By correctly accounting for the differential reliability of observations, WLS produces coefficient estimates that have the minimum possible variance among all linear unbiased estimators. This statistical efficiency translates directly into more precise parameter estimates, narrower confidence intervals, and increased statistical power, ultimately enhancing the reliability and robustness of the resulting scientific conclusions. Moreover, unlike post-estimation corrections like HC standard errors, WLS modifies the estimation process itself, ensuring that the model fit is optimally calibrated according to the true underlying structure of the data variances.

However, WLS is subject to significant practical and methodological limitations, primarily centered on the difficulty of accurately defining the weights. As previously detailed, if the weights are based on an estimated variance function (FWLS), the estimation of this function itself introduces potential error, which can propagate through the system. Misspecification of the weights, either through choosing the wrong functional form or through measurement error in the proxy variables used for weighting, can lead to estimates that are less efficient, or even biased, relative to a simple OLS approach. This inherent reliance on accurate weight specification increases the complexity and uncertainty associated with WLS application in exploratory research contexts.

Furthermore, WLS can be highly sensitive to outliers in the data, particularly if those outliers heavily influence the estimation of the variance function used to calculate the weights. An atypical observation might artificially inflate or deflate the estimated variance structure in its neighborhood, thereby distorting the calculated weights for a range of observations across the dataset. The methodological complexity involved in accurately modeling the variance function means WLS requires a substantial degree of statistical intuition, domain-specific knowledge, and rigorous diagnostic efforts compared to the straightforward application of OLS. Researchers must always ensure that the use of weights is clearly justified by the scientific context and not merely employed as a statistical manipulation tool.

7. Applications Across Disciplines

WLS is a highly versatile and indispensable statistical tool employed across numerous quantitative fields whenever data variance is expected to scale systematically with the magnitude of the measured variable or is linked to the precision of the sampling mechanism. In **Econometrics and Finance**, WLS is frequently applied to analyze time-series data where volatility (variance) is known to change over time (conditional heteroscedasticity), or in cross-sectional analyses dealing with heterogeneous entities, such as firms or countries, where larger entities naturally exhibit greater variance in metrics like revenue or investment compared to smaller ones.

In **Metrology, Chemistry, and Physics**, WLS is used extensively in calibration and measurement

science. When measuring concentrations or quantities using instruments, the precision of the measurement often systematically decreases as the measured value increases, leading to larger errors at higher concentrations. WLS provides the methodology to properly combine these measurements by assigning lower weights to the less precise, high-value measurements, ensuring optimal parameter estimates for the fundamental calibration curves used to relate instrument signals to actual quantities.

In **Biostatistics and Public Health**, WLS is crucial when analyzing aggregated data or when combining results from multiple independent studies, a process known as **meta-analysis**. In meta-analysis, individual study results are weighted by the inverse of their estimated variance (i.e., by their precision), ensuring that studies with larger sample sizes and more accurate estimates contribute more heavily to the combined overall effect size--a direct and powerful application of the WLS principle to synthesize scientific evidence. Similarly, in complex survey research, WLS can be used to account for sampling designs where certain population segments are intentionally over- or under-sampled, necessitating the use of sampling weights to ensure the regression estimates accurately reflect the parameters of the true population.

Further Reading

[Weighted least squares - Wikipedia](#)

[Ordinary least squares - Wikipedia](#)

[Heteroscedasticity - Wikipedia](#)

[Breusch-Pagan test - Wikipedia](#)

[Generalized least squares - Wikipedia](#)