

# WATER-JUG PROBLEMS

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## WATER-JUG PROBLEMS

**Primary Disciplinary Field(s):** Cognitive Psychology; Mathematics; Computer Science (Artificial Intelligence)

### 1. Core Definition

The Water-Jug Problems constitute a classic class of puzzles and mathematical challenges employed extensively across cognitive science and computer science, primarily serving as a paradigm for studying human and artificial problem-solving capabilities. These problems invariably require the actor--be it a human subject or an intelligent algorithm--to determine a precise, specified quantity of liquid using only a restricted set of containers (jugs) whose capacities are fixed and known. The underlying constraints dictate that the only permissible actions are filling a jug completely, emptying a jug completely, or pouring water from one jug into another until either the source jug is empty or the destination jug is full. The defining feature is the necessity of achieving a target volume that cannot be directly measured by any single jug, thus demanding a sequence of non-trivial operations. For instance, a common variant might ask how to measure exactly 4 liters of water when provided only with a 5-liter jug and a 3-liter jug, requiring multiple transfers to reach the desired intermediate state. These problems are fundamentally characterized by a well-defined initial state, a clearly specified goal state, and a finite set of operators (rules) governing transitions between states, making them an ideal framework for analyzing search strategies and cognitive efficiency.

The psychological utility of the Water-Jug Problem stems from its simplicity and the quantifiable nature of the steps required for resolution, allowing researchers to meticulously track the decision-making process. Unlike real-world problems which often involve ambiguous goals or ill-defined operators, the constraints in the jug problem are absolute, permitting rigorous analysis of errors, heuristic deployment, and response latency. The core task involves navigating a complex state space--the theoretical construct encompassing all possible combinations of water levels in the given jugs--to find the shortest or most efficient path from the starting configuration (usually all jugs empty) to the goal configuration (one jug containing the target volume). This state-space search requires subjects to engage in both forward-thinking (planning moves toward the goal) and sometimes backward-thinking (working backward from the goal state), demanding significant working memory capacity and the ability to overcome initial perceptual biases related to the jug capacities themselves.

Crucially, the problem set emphasizes the concept of intermediate measurements; the solution is rarely direct but relies on utilizing the fixed capacities of the jugs to subtract or add volumes indirectly. Consider the problem where the target quantity is 39 ml, but the available jugs are 207 ml, 165 ml, and 42 ml. Achieving 39 ml requires skillful manipulation of these large volumes to

isolate the target difference, often involving filling the largest jug and then using the smaller jugs to iteratively remove specific quantities. This process underscores the necessity of recursive thinking and the ability to maintain multiple intermediate goal states simultaneously. While physical jugs are rarely supplied in testing environments--the problem is typically presented abstractly or visually on paper or screen--the mental simulation required remains robust, testing the participant's executive functions related to planning and inhibition of impulsive, non-optimal moves.

## 2. Mathematical Foundation

Mathematically, the solvability of a Water-Jug Problem is determined by principles rooted in number theory, specifically concerning the Greatest Common Divisor (GCD) and Bézout's identity. If two jugs have capacities  $A$  and  $B$ , the only measurable volumes achievable are integer linear combinations of  $A$  and  $B$ . That is, any volume  $V$  that can be measured must be expressed as  $V = xA + yB$ , where  $x$  and  $y$  are integers (representing the number of times a jug is filled or emptied, positively or negatively). According to Bézout's identity, the smallest positive quantity that can be measured is exactly equal to  $\text{GCD}(A, B)$ . Therefore, a fundamental solvability criterion is established: if the target quantity  $T$  is not a multiple of the GCD of the capacities of all jugs involved, the problem is mathematically impossible to solve. This criterion provides a definitive means of assessing the inherent feasibility of any given Water-Jug Problem configuration before attempting a search for a solution path.

For problems involving two jugs, the relationship simplifies to a form of a linear Diophantine equation. The state space can be visualized as a grid where coordinates represent the volume in each jug, and the transitions (pouring, filling, emptying) define the possible movements across this grid. An optimal solution corresponds to the shortest path through this space. The sequence of operations can be modeled efficiently using algorithms derived from graph theory, where each attainable state is a node and each valid move is a directed edge. The mathematical elegance of the problem lies in the fact that despite the combinatorial explosion of the potential state space, the underlying solution structure is dictated by relatively simple modular arithmetic. Furthermore, the problem often exhibits certain invariant properties, such as the total amount of water remaining constant during a transfer operation, which helps constrain the search space and allows for the application of computational optimization techniques.

When three or more jugs are introduced, the complexity increases, but the dependency on the GCD remains paramount. If a target volume  $T$  must be measured using three jugs with capacities  $A$ ,  $B$ , and  $C$ , the target must still be divisible by  $\text{GCD}(A, B, C)$ . This mathematical rigor allows computer scientists to model the problem precisely. The focus shifts from merely finding a solution to finding the most efficient solution, often defined as the minimum number of steps or transfers required. This pursuit of optimality drives the application of advanced search algorithms, such as Dijkstra's algorithm or A\* search, which use heuristics (estimates of the

remaining cost) to prune inefficient branches of the solution tree, significantly reducing computational overhead compared to brute-force methods. The mathematical representation ensures that the problem serves as a perfect analogue for testing the efficiency and completeness of such search methodologies in artificial intelligence.

### 3. Historical Context: Luchins and the Einstellung Effect

While the origins of water-pouring puzzles date back to ancient mathematical texts--including possible references in the work of Niccolò Fontana Tartaglia--the Water-Jug Problems gained profound prominence in the field of psychology due to the seminal work conducted by Abraham S. Luchins in the 1940s. Luchins utilized a specific series of these problems to investigate the concept of "Einstellung," a German term roughly translating to "set" or "attitude." His experiments demonstrated how prior successful experience with a specific solution method could severely impede a subject's ability to find a simpler, more direct solution to a subsequent, slightly different problem. This phenomenon is now widely recognized as the Einstellung effect or the mechanization of thought.

In the classic Luchins experiment, participants were presented with a series of problems (Jugs 1 through 5) that all required a complex, three-step formula (e.g., Fill B, pour B into A until A is full, empty A, pour remaining B into A, etc., summarized as  $B - A - 2C$ ) to achieve the target volume. After successfully solving these training problems using this complex method, subjects were then given critical test problems (Jugs 6 and 7). These test problems could also be solved using the established complex  $B-A-2C$  method, but they also possessed much simpler, two-step solutions ( $A-C$  or  $A+C$ ). Luchins observed a powerful demonstration of mental inertia: the majority of subjects continued to apply the cumbersome, multi-step method they had learned, failing to recognize or employ the simpler, more efficient solution available.

This reluctance to shift strategies, even when a more optimal path is immediately available, vividly illustrates the psychological concept of mental set or cognitive rigidity. The successful repetition of the initial, complex procedure created a strong mental predisposition--the Einstellung--that blinded the participants to alternative, superior solutions. This finding was critically important to the understanding of transfer of learning and problem-solving heuristics. It demonstrated that experience, while usually beneficial, can sometimes become a liability, locking the solver into an inefficient routine. The results provided robust empirical evidence contradicting simpler models of problem solving that focused purely on rational decision-making, highlighting instead the powerful, often unconscious influence of past methods on current thought processes.

The enduring significance of Luchins' work lies in its contribution to understanding fixation and cognitive bias. The Water-Jug Problems, through the demonstration of the Einstellung effect, became a cornerstone in the study of productive versus reproductive thinking. Productive thinking

involves genuinely novel restructuring of the problem space, while reproductive thinking involves merely applying previously learned methods. The subjects applying the complex B-A-2C formula were engaging in reproductive thinking, failing the test of cognitive flexibility. This paradigm continues to be used today to explore individual differences in susceptibility to set effects, the role of motivation and time pressure, and the impact of explicit instructions aimed at breaking the established mental pattern.

#### 4. Modeling in Artificial Intelligence

The inherent structure of the Water-Jug Problems--characterized by discrete states, defined operators, and a clear goal--makes them an ideal benchmark for testing algorithms in Artificial Intelligence (AI) and automated planning. In AI, the problem is formalized as a state-space search task. Each state is defined by an ordered tuple representing the current volume of water in every jug (e.g., for two jugs (A, B), a state might be (a, b)). The goal state is any tuple where one of the components equals the target volume T. The operators--fill, empty, pour--become the transition rules that connect one state to another.

AI systems approach this problem by constructing a graph where nodes are states and edges are valid operations. The primary objective is to find a sequence of edges (the solution path) leading from the initial state (e.g., (0, 0)) to the goal state. Standard search algorithms are employed to navigate this graph. Breadth-First Search (BFS) guarantees finding the shortest solution path (the optimal number of steps) by systematically exploring all states one step away from the start, then two steps away, and so forth. However, for problems with vast state spaces, BFS can be computationally expensive as it explores unnecessary branches.

Alternatively, Depth-First Search (DFS) explores as deep as possible along a single path before backtracking, which is memory efficient but does not guarantee optimality. For more complex planning and optimization, heuristic search algorithms like A\* search are often preferred. A\* uses an evaluation function that combines the cost already incurred to reach the current state with a heuristic estimate of the cost remaining to reach the goal. Effective heuristics, such as estimating the remaining volume difference that needs to be generated, dramatically speed up the search by guiding the algorithm toward the most promising states. The Water-Jug Problem serves as an excellent pedagogical tool in AI education for illustrating the tradeoffs between algorithm completeness, optimality, and computational efficiency.

#### 5. Solution Strategies and Algorithms

Human solvers and computational algorithms employ distinct yet sometimes overlapping strategies to tackle Water-Jug Problems. Human strategies often involve a mix of trial-and-error, mental simulation, and the application of generalized heuristics. A crucial human strategy is the application

of difference reduction: recognizing that the solution involves generating the target quantity by finding the appropriate difference between the jug capacities, and then working to isolate that difference. When faced with a novel problem, subjects frequently engage in forward search, starting from the initial state and attempting to move closer to the goal. However, expert solvers or those trained in sophisticated problem-solving techniques may also utilize backward chaining, starting from the goal state and determining the preceding state(s) from which the goal could have been achieved, reversing the process until the initial state is reached.

Computational solutions prioritize finding the mathematically optimal path. The state-space approach formalized in AI allows for the systematic enumeration of all reachable states. For the two-jug case, specific algebraic algorithms can bypass general graph search entirely. Since the problem involves modular arithmetic, specialized algorithms can calculate the precise sequence of fills and pours required to achieve the required linear combination defined by Bézout's identity. These specialized number-theoretic approaches are extremely efficient but lose their generality when the constraints of the problem change significantly (e.g., allowing for spillage or introducing jugs of unknown capacity).

The contrast between human and algorithmic strategies highlights the difference between psychological and computational optimality. While an algorithm seeks the absolute minimum number of steps, human solvers are constrained by cognitive load. Therefore, a human solution might prioritize moves that maintain a clear, intermediate sub-goal state (e.g., getting exactly 1 liter in the smallest jug) rather than following a sequence of moves that, while mathematically shorter, might be harder to conceptualize or remember. This divergence underscores the psychological relevance of the problems, demonstrating that ease of comprehension and working memory constraints often supersede pure numerical efficiency in human problem-solving contexts.

## 6. Key Characteristics

**Closed Problem Space:** The problem is highly constrained; the initial state, goal state, and available operators (fill, empty, pour) are all finite and explicitly defined, classifying it as a "well-defined problem" in cognitive science.

**Dependency on GCD:** Solvability is fundamentally reliant on number theory, specifically the relationship between the target volume and the Greatest Common Divisor of the jug capacities. If the target is not a multiple of the GCD, the problem has no solution.

**Tool for Measuring Cognitive Rigidity:** Historically, the problems are most famous for their utility in demonstrating the Einstellung effect (mental set), where prior successful methods impede the discovery of simpler solutions.

**State-Space Search Paradigm:** The structure perfectly maps onto graph theory, allowing it to be

modeled computationally as a search for the shortest path in a defined state graph, a cornerstone application in introductory AI.

## 7. Significance and Impact

The enduring significance of the Water-Jug Problems spans mathematics, psychology, and computer science. In cognitive psychology, their impact is monumental, serving as the foundational tool for studying cognitive set, mental rigidity, and the mechanisms of transfer of learning since the 1940s. They provide a standardized, replicable experimental framework that allows researchers to isolate specific aspects of human thought, such as planning, inhibition, and the transition from reproductive to productive problem-solving strategies. The simplicity of the apparatus belies the complexity of the mental operations required, making them invaluable for understanding human errors and biases under defined constraints.

In mathematics, the problems serve as a tangible illustration of abstract number-theoretic concepts, particularly Diophantine equations and Bézout's identity. They bridge pure mathematical theory with practical, though stylized, application. Furthermore, the problem structure offers a rich source of combinatorial analysis, inviting exploration into algorithms designed to find optimal solutions across varying numbers of jugs and capacity configurations. The mathematical foundation ensures that the problem remains a relevant topic in recreational mathematics and educational curricula focused on logic and abstract reasoning.

For computer science and artificial intelligence, the Water-Jug Problems are a canonical example of a generic search problem. They are often the first practical exercise introduced when teaching students about state-space representation, graph traversal algorithms (BFS, DFS), and heuristic search methods (A\*). Because the problem is finite and deterministic, it allows for clear comparison between different search methodologies and performance optimization, establishing a fundamental paradigm for understanding how intelligent agents navigate complex decision trees to achieve specific goals efficiently.

## Further Reading

[Water Pouring Puzzle - Wikipedia](#)

[Einstellung Effect - Wikipedia](#)

[Luchins, A. S. \(1942\). Mechanization in problem solving: The effect of Einstellung. Psychological Monographs, 54\(6, Whole No. 248\).](#)

[State Space Search - Wikipedia](#)