What is Error Propagation?

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Error propagation is a mathematical concept that describes how uncertainties in measured quantities can affect the accuracy of calculated results. It involves tracing the effects of these uncertainties through a series of mathematical operations, such as addition, subtraction, multiplication, and division, to determine the overall uncertainty in the final result. This concept is crucial in scientific and engineering fields, as it allows for a better understanding of the reliability and precision of experimental data and calculations. By identifying and quantifying the sources of error, error propagation helps to improve the accuracy and credibility of scientific findings.

What is Error Propagation? (Definition & Example)

Error propagation occurs when you measure some quantities *a*, *b*, *c*, ... with uncertainties δ*a*, δ*b*, δ*c* ... and you then want to calculate some other quantity *Q* using the measurements of *a*, *b*, *c*, etc.

It turns out that the uncertainties δ*a*, δ*b*, δ*c* will propagate (i.e. "extend to") to the uncertainty of *Q*.

To calculate the uncertainty of *Q*, denoted δ*Q*, we can use the following formulas.

Note: For each of the formulas below, it's assumed that the quantities *a*, *b*, *c*, etc. have errors that are *random* and *uncorrelated*.

Addition or Subtraction

If $Q = a + b + ... + c - (x + y + ... + z)$

Then δ*Q* = √(δa)2 + (δb)2 + ... + (δc)2 + (δx)2 + (δy)2 + ... + (δz)2

Example: Suppose you measure the length of a person from the ground to their waist as 40 inches ± .18 inches. You then measure the length of a person from their waist to the top of their head as 30 inches ± .06 inches.

Suppose you then use these two measurements to calculate the total height of the person. The height would be calculated as 40 inches + 30 inches = 70 inches. The uncertainty in this estimate would be calculated as:

δ*Q* = √(δa)2 + (δb)2 + ... + (δc)2 + (δx)2 + (δy)2 + ... + (δz)2δ*Q* = √(.18)2 + (.06)2δ*Q* = 0.1897

This gives us a final measurement of 70 ± 0.1897 inches.

Multiplication or Division

If *Q* = (ab...c) / (xy...z)

Then δ*Q* = |Q| * √(δa/a)2 + (δb/b)2 + ... + (δc/c)2 + (δx/x)2 + (δy/y)2 + ... + (δz/z)2

Example: Suppose you want to measure the ratio of the length of item *a* to item *b*. You measure the length of *a* to be 20 inches ± .34 inches and the length of *b* to be 15 inches ± .21 inches.

The ratio defined as *Q* = *a*/*b* would be calculated as 20/15 = 1.333. The uncertainty in this estimate would be calculated as:

$\delta Q = |Q| * \sqrt{(\delta a/a)^2 + (\delta b/b)^2 + \ldots + (\delta c/c)^2 + (\delta x/x)^2 + (\delta y/y)^2 + \ldots + (\delta z/z)^2}$ $\delta Q = |1.333| * \sqrt{(.34/20)^2 + (.21/15)^2}$ $\delta Q = 0.0294$

Measured Quantity Times Exact Number

If *A* is known exactly and *Q* = *A*x

Then $\delta Q = |A|\delta x$

Example: Suppose you measure the diameter of a circle as 5 meters ± 0.3 meters. You then use this value to calculate the circumference of the circle *c* = π*d*.

The circumference would be calculated as *c* = π*d* = π*5 = 15.708. The uncertainty in this estimate would be calculated as:

$\delta Q = |A|\delta x \delta Q = |\pi| * 0.3 \delta Q = 0.942$

Thus, the circumference of the circle is 15.708 ± 0.942 meters.

Uncertainty in a Power

If *n* is an exact number and *Q* = *xn*

Then $\delta Q = |Q| * |n| * (\delta x/x)$

Example: Suppose you measure the side of a cube to be *s* = 2 inches ± .02 inches. You then use this value to calculate the volumne of the cube *v* = *s*3.

The volume would be calculated as *v* = *s*3 = 23 = 8 in.3. The uncertainty in this estimate would be calculated as:

$\delta Q = |Q| * |n| * (\delta x/x) \delta Q = |8| * |3| * (.02/2) \delta Q = 0.24$

Thus, the volume of the cube is 8 ± .24 in.3.

General Formula for Error Propagation

If *Q* = *Q(x)* is any function of *x* then the general formula for error propagation can be defined as:

$\delta Q = |dQ/dX|\delta x$

Note that you'll rarely have to derive these formulas from scratch, but it can be good to know the general formula used to derive them.