what is Error Propagation? (Definition & Example)

Authored by

stats writer

December 10, 2025

RECOMMENDED CITATION

stats writer (2025). *what is Error Propagation? (Definition & Example)*. PSYCHOLOGICAL SCALES. Retrieved from https://scales.arabpsychology.com/?p=107037

Error propagation is a fundamental concept in experimental science and engineering, defining the methodology for determining how the inherent uncertainties associated with measured input quantities influence the final calculated result. Whenever physical measurements are taken, they are subject to measurement errors; these initial errors do not simply vanish when combined mathematically--they are carried forward, or propagated, into the derived value.

For instance, if you measure the dimensions of a physical object, both the length and width will have associated measurement errors. When these two values are multiplied to calculate the area, the error from the length and the error from the width combine, resulting in an overall uncertainty in the final area calculation. Understanding this mechanism is critical for rigorously reporting and interpreting scientific data, ensuring that the reliability of the derived quantity is accurately quantified based on the reliability of the inputs.

Error propagation occurs whenever a derived quantity Q is calculated using a set of measured variables $a, b, c, dots$, each associated with its own absolute uncertainty ($delta a, delta b, delta c, dots$). The process involves estimating how these input errors combine statistically to determine the final uncertainty of the calculated result, denoted as $delta Q$.

The uncertainties $delta a, delta b, delta c$ will invariably **propagate** and contribute to the resulting uncertainty of Q. To accurately calculate $delta Q$, specific mathematical formulas are utilized, which depend directly on the functional relationship between the input variables and the quantity Q.

The Statistical Foundation of Error Propagation

The standard formulas used for error propagation are built upon stringent statistical assumptions concerning the nature of the errors in the measured quantities. It is vital to recognize these assumptions, as deviating from them requires more complex computational methods.

The most fundamental assumption is that the quantities $a, b, c,$ etc., have errors that are **random**. A random error implies that fluctuations in the measurement process are unpredictable and are likely to scatter equally above and below the true value, often characterized by a normal distribution. Systematic errors, such as a calibration flaw, must be corrected for before applying these formulas, as systematic bias cannot be modeled using standard error propagation.

Secondly, the errors are assumed to be **uncorrelated**. This means that the error in measuring variable a is statistically independent of the error in measuring variable b. If the measurements are dependent (e.g., if one measurement influences the environment for the next, or if two variables are derived from the same initial faulty reading), a covariance term must be introduced, which complicates the calculation significantly. The simplified formulas presented here are strictly valid only for independent, uncorrelated measurements.

Error Propagation for Sums and Differences

When the final quantity Q is determined by adding or subtracting the measured variables, the absolute uncertainties of the input variables combine in quadrature. This method, involving the square root of the sum of the squared absolute errors, prevents the final uncertainty from being grossly overestimated, reflecting the statistical likelihood that some random errors will partially offset others.

If the functional relationship is defined as a linear combination of inputs:

If **$Q = a + b + dots + c - (x + y + dots + z)$**

Then the uncertainty $delta Q$ is calculated using the following structure. Note that the sign of the operation (addition or subtraction) does not affect how the uncertainties combine; they are always added in quadrature:

Then **$delta Q = sqrt{left(delta aright)^{2} + left(delta bright)^{2} + dots + left(delta cright)^{2} + left(delta xright)^{2} + left(delta yright)^{2} + dots + left(delta zright)^{2}}$**

Practical Example: Combining Length Measurements

Example: Suppose a researcher measures the length of a person from the ground to their waist as $a = 40 text{ inches } pm 0.18 text{ inches}$. They then measure the length from the waist to the top of their head as $b = 30 text{ inches } pm 0.06 text{ inches}$. These measurements are used to calculate the total height, Q.

The total height calculation is $Q = 40 text{ inches} + 30 text{ inches} = mathbf{70}$ inches. The uncertainty in this estimate, where $delta a = 0.18$ and $delta b = 0.06$, would be calculated as:

$delta Q = sqrt{(delta a)^{2} + (delta b)^{2}}$
$delta Q = sqrt{(0.18)^{2} + (0.06)^{2}}$
$delta Q approx mathbf{0.1897}$

This yields a final measurement of $mathbf{70 pm 0.1897}$ inches. This example demonstrates how the measurement with the largest absolute uncertainty (0.18 inches) often dictates the overall precision of the final sum.

Error Propagation for Products and Quotients

When the relationship between variables involves multiplication or division, the method of error combination shifts from absolute errors to **fractional** or **relative errors**. Relative error is defined as the ratio of the absolute uncertainty to the measured value ($delta x / x$). This approach is necessary because the magnitude of the resulting error is proportional to the overall size of the calculated quantity Q.

For a quantity Q defined by multiplicative and divisive operations:

If **$Q = frac{(a cdot b cdot dots cdot c)}{(x cdot y cdot dots cdot z)}$**

The fractional uncertainties of all inputs are combined in quadrature, and the resulting combined fractional uncertainty is then multiplied by the absolute value of Q to convert back to the absolute uncertainty $delta Q$:

Then **$delta Q = |Q| cdot sqrt{left(frac{delta a}{a}right)^{2} + left(frac{delta b}{b}right)^{2} + dots + left(frac{delta c}{c}right)^{2} + left(frac{delta x}{x}right)^{2} + left(frac{delta y}{y}right)^{2} + dots + left(frac{delta z}{z}right)^{2}}$**

Practical Example: Calculating a Ratio

Example: Suppose we want to measure the ratio of the length of item a to item b. Length a is measured as $20 text{ inches} pm 0.34 text{ inches}$, and length b is measured as $15 text{ inches} pm 0.21 text{ inches}$. The derived quantity is $Q = a/b$.

The primary calculated ratio is $Q = 20/15 approx mathbf{1.333}$. To find the uncertainty $delta Q$, we calculate the fractional errors and apply the formula:

$delta Q = |Q| cdot sqrt{left(frac{delta a}{a}right)^{2} + left(frac{delta b}{b}right)^{2}}$
$delta Q = |1.333| cdot sqrt{left(frac{0.34}{20}right)^{2} + left(frac{0.21}{15}right)^{2}}$
$delta Q approx mathbf{0.0294}$

The resulting ratio is $mathbf{1.333 pm 0.0294}$. This methodology is indispensable when calculating physical quantities like density (mass divided by volume), where errors in multiple proportional inputs must be accurately propagated.

Dealing with Measured Quantity Times Exact Number

A simplified case of multiplication occurs when one factor in the calculation is a constant known exactly, A, meaning its uncertainty is effectively zero. Examples of exact numbers include mathematical constants like pi or conversion factors.

If **A is known exactly and $Q = A cdot x$**

Since the exact number A contributes no error itself, it serves only to scale the existing uncertainty of the measured variable x. The formula simplifies to a straightforward multiplication:

Then **$delta Q = |A| cdot delta x$**

Practical Example: Scaling Circumference

Example: Suppose the diameter d of a circle is measured as $5 text{ meters} pm 0.3 text{ meters}$. We use this value to calculate the circumference $c = pi d$. Here, $A = pi$ (exact) and $x = d = 5 pm 0.3$.

The calculated circumference is $c = pi cdot 5 approx mathbf{15.708}$ meters. The uncertainty $delta Q$ is found by scaling the measured uncertainty by pi:

$delta Q = |A| cdot delta x$
$delta Q = |pi| cdot 0.3$
$delta Q approx mathbf{0.942}$

Thus, the circumference is reported as $mathbf{15.708 pm 0.942}$ meters. The absolute error is linearly magnified by the constant pi.

Uncertainty in a Power

When a quantity Q depends on a single measured variable x raised to an exact numerical power n (such as calculating volume or area), this is treated as a specialized case of the multiplication rule. This propagation method is critical because raising a variable to a power significantly amplifies the relative error.

If **n is an exact number and $Q = x^{n}$**

The uncertainty $delta Q$ is found by relating the fractional error of Q to the fractional error of x, scaled by the exponent n:

Then **$delta Q = |Q| cdot |n| cdot left(frac{delta x}{x}right)$**

Practical Example: Calculating Cube Volume Uncertainty

Example: Suppose the side length s of a cube is measured as $2 text{ inches } pm 0.02 text{ inches}$. We calculate the volume $v = s^3$. Here, $x=s$ and $n=3$.

The calculated volume is $v = 2^3 = mathbf{8 text{ in.}^{3}}$. The uncertainty in this estimate would be calculated as:

$delta Q = |Q| cdot |n| cdot left(frac{delta x}{x}right)$

$delta Q = |8| cdot |3| cdot left(frac{0.02}{2}right)$
$delta Q = mathbf{0.24}$

Thus, the volume of the cube is $mathbf{8 pm 0.24 text{ in.}^{3}}$. The fractional error in the volume ($0.24/8 = 3%$) is three times the fractional error in the side length ($0.02/2 = 1%$).

The Generalized Formula for Error Propagation

The specific rules discussed above are derived from a powerful, fundamental principle applicable to any function of measured variables. This general approach uses calculus, specifically the law of propagation of errors based on partial derivatives.

If Q is any function of independent variables $x_1, x_2, dots, x_N$, the general formula for error propagation is defined as:

$delta Q = sqrt{ left(frac{partial Q}{partial x_1} delta x_1 right)^2 + left(frac{partial Q}{partial x_2} delta x_2 right)^2 + dots + left(frac{partial Q}{partial x_N} delta x_N right)^2 }$

For the simplest case where $Q = Q(x)$ is a function of a single variable x, this simplifies to the absolute value of the derivative multiplied by the uncertainty $delta x$:

$delta Q = left|frac{dQ}{dx}right| cdot delta x$

Note that you will rarely have to derive these formulas from scratch in routine laboratory work, as the specific rules are highly reliable. However, knowing the general formula involving the partial derivative is essential for handling complex, non-linear relationships or when the measurement uncertainties are reported as the standard deviation. This methodology ensures robust calculation of the final result's reliability.

Summary of Essential Formulas

For quick reference, here is a concise recap of the primary rules for error propagation, assuming inputs are **random** and independent:

Addition or Subtraction: If $Q = a pm b$, then $delta Q = sqrt{(delta a)^{2} + (delta b)^{2}}$
Multiplication or Division: If $Q = a cdot b$ or $Q = a/b$, then $delta Q = |Q| cdot sqrt{left(frac{delta a}{a}right)^{2} + left(frac{delta b}{b}right)^{2}}$
Scaling by Constant: If $Q = A cdot x$ (where A is exact), then $delta Q = |A| cdot delta x$
Uncertainty in a Power: If $Q = x^{n}$ (where n is exact), then $delta Q = |Q| cdot |n| cdot left(frac{delta x}{x}right)$